COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 8 (4) 1988 345—356

LIMIT THEOREM CONCERNING RANDOM MAPPING PATTERNS

L. R. MUTAFCHIEV

Received July 24, 1986
Revised August 28, 1987

Mapping patterns may be represented by unlabelled directed graphs in which each point has out-degree one. Assuming uniform probability distribution on the set of all mapping patterns on n points, we obtain limit distributions of some characteristics associated with the graphs of mapping patterns (connected and disconnected), as $n\to\infty$. In particular, we study the number of points belonging to cycles, the number of cycles and components having prescribed (fixed) number of points and the total number of components.

1. Introduction

Let f be a single-valued mapping of the set $N=\{1, 2, ..., n\}$ into itself. The function f may be represented by a directed graph G_f with pointset N and arc-set $\{(i,f(i)):\ i\in N\}$. Thus, every point of G_f has out-degree one and G_f consists of a number of disjoint components with exactly one cycle in each. The points of G_f may be considered as lying in directed trees whose roots coincide with the cyclic points; the arcs of each tree are directed towards the root of the tree. Sometimes these graphs are called functional digraphs [6; Chap. 3, §4]. A probability measure may be introduced on the set of all mappings f (or on the set of the functional digraphs G_f). In this way all conceivable numerical characteristics of G_f become random variables. There are fairly many results describing the exact distributions of various random variables, associated with the graphs G_f, and their asymptotic behaviour, as $n\to\infty$ (see e.g. the monograph [9] or the survey paper [12]).

Two functions f and g, as defined above, are said to be equivalent if there exists a permutation π of N such that $f(i)=j$ iff $g(\pi(i))=\pi(j)$ for all $i\in N$. An equivalence class of mapping functions is called a mapping pattern. Thus, a mapping pattern may be considered as a functional digraph on n unlabelled points. Denote by p_n the number of mapping patterns on n points. Important steps for determining the numbers p_n were made by Harary [5] and Read [14]. A formal expression for the generating function $P(x)=1+\sum_{n=1}^{\infty} p_n x^n$ was obtained in [5] in temrs of the generating function $T(x)=\sum_{n=1}^{\infty} t_n x^n$, where t_n denotes the number of rooted trees with n unlabelled points. The expression for $P(x)$ was simplified in [14]. (For historical comments

AMS subject classification code (1980): 60 C 05, 05 C 30.

about the problem of determining the numbers p_n and a combinatorial derivation of Read's simplification see also [2].) Meir and Moon [11] carried out an asymptotic analysis of the coefficients of the corresponding generating functions and obtained the asymptotic behaviour of various parameters associated with mapping patterns. After having determined the asymptotics of p_n and of the numbers c_n of all connected mapping patterns on n points as $n\to\infty$, they studied also the expected number of points belonging to cycles (the expectations are taken over p_n and c_n mapping patterns) and the expected number of components (the expectation is taken over p_n mapping patterns).

Our purpose here is to obtain information about the asymptotic behaviour of such characteristics of mapping patterns in terms of limit distributions. Section 2 contains auxiliary and known facts of combinatorial and analytical character we shall need later. Throughout the paper we assume that each mapping pattern occurs with probability $1/p_n$ or $1/c_n$. We denote the corresponding probability measures by $P^{(p_n)}$ and $P^{(c_n)}$, respectively. Under these assumptions we prove, in Section 3, two local limit theorems for the number of cyclic points in a random mapping pattern with respect to these two different probability measures. In Section 4 limit theorems for the number of components and cycles with prescribed (fixed) number of points are proved. In Section 5 we obtain an integral limit theorem for the total number of components of a random mapping pattern. A comarison beween these limit distributions and the known results for random mapping functions in which the labels of points are taken into account is given in Section 6. Table 1 in the next Section makes easier the immediate understanding of this comparison.

2. Preliminary results

As it was mention in the Introduction we shall deal with two different probability measures. In view of this we shall denote by $E^{(p_n)}(\cdot)$ and $E^{(c_n)}(\cdot)$ the expectations taken over p_n and c_n mapping patterns, respectively.

Let $F(x)$ be any power series. We shall use the notations $Z(C_k, F(x))$ and $Z(S_r, F(x))$ for the power series obtained by substituting $F(x)$ into the cycle index of the cyclic group C_k and the symmetric group S_r, respectively. It is wellknown that

$$Z(C_k, F(x)) = \frac{1}{k}\sum_{d|k}\varphi(d)F(x^d))^{k/d}, \tag{2.1}$$

where φ denotes the Euler phi-function, and that

$$1+\sum_{r=1}^{\infty} Z(S_r, F(x))z^r = \exp\sum_{r=1}^{\infty} z^r F(x^r)/r \tag{2.2}$$

for any variable z. (The definition of the cycle index of a permutation group and the connection between cycle-indeces and Pólya enumeration theorem may be found e.g. in [6; Chap. 2]).

Let α_k be the number of cycles with k points, and let β_k be the number of components with k points of a random mapping pattern, $k=1, 2, \ldots$. The first relation of the following lemma is contained in [5]; the second relation follows analogously.

Table 1

Characteristic	Limit distributions for labelled functional digraphs	Limit distributions for unlabelled functional digraphs (mapping patterns)	Explanatory notes
The number of cyclic points in a mapping/$n^{1/2}$	$1-e^{-x^2/2}$, $0<x<\infty$ (see [7]).	$1-e^{-cx^2/2}$, $0<x<\infty$ (see Theorem 1).	$c=b^2\varrho/2=1.2159\ldots$; see (2.4) and (2.5)
The number of cyclic points in a connected mapping/$n^{1/2}$	$(2/\pi)^{1/2}\int_0^x e^{-u^2/2}du$, $0<x<\infty$ see [15].	$(2c/\pi)^{1/2}\int_0^x e^{-cu^2/2}du$, $0<x<\infty$ (see Theorem 2)	$c=b^2\varrho^{1/2}/2=$ $=1.2159\ldots$; see (2.4) and (2.5).
The number of cycles on j points	The coefficient of y^k in $e^{(y-1)/j}$, $k=0, 1, \ldots$ (see [16]).	The coefficient of y^k in $\exp \sum_{m=1}^{\infty} d_m(j)(y^m-1)/m$, $k=0, 1, \ldots$ (see Theorem 3)	$d_m(j)=Z(C_j,T(\varrho^m))$ see (2.1), (2.4) and (2.5)
The number of components on j points	The coefficient of y^k in $e^{\mu_j(y-1)}$, $\mu_j=(e^{-j}/j)\sum_{m=0}^{j-1} j^m/m!$, $k=0, 1, \ldots$ (see [8]).	The coefficient of y^k in $(1-\varrho^j)^{c_j}(1-\varrho^j y)^{-c_j}$, $k=0, 1, \ldots$ (see Theorem 4)	See (2.4); c_j denotes the number of all connected mapping patterns on points.
(The total number of components $-\frac{1}{2}\log n\Big)\Big/\Big(\frac{1}{2}\log n\Big)^{1/2}$	$(2\pi)^{-1/2}\int_{-\infty}^x e^{-u^2/2}\,du$, $-\infty<x<\infty$ (see [16]).	$(2\pi)^{-1/2}\int_{-\infty}^x e^{-u^2/2}\,du$, $-\infty<x<\infty$ (see Theorem 5)	

Lemma 1. *Let $x, y_1, y_2, \ldots$ be any variables satisfying the inequalities $|x| \leqq \varrho$, $|y_k| \leqq 1$, where ϱ is the radius of convergence of the generating function $T(x)$ of the rooted unlabelled trees. Then*

$$\text{(i)} \qquad \prod_{k=1}^{\infty} \exp \sum_{m=1}^{\infty} \frac{y_k^m}{m} Z(C_k, T(x^m)) = 1 + \sum_{n=1}^{\infty} p_n x^n E^{(p_n)}\Big(\prod_{k=1}^{n} y_k^{\alpha_k}\Big);$$

$$\text{(ii)} \qquad \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=1}^{\infty} c_k y_k^m x^{km} = 1 + \sum_{n=1}^{\infty} p_n x^n E^{(p_n)}\Big(\prod_{k=1}^{n} y_k^{\beta_k}\Big).$$

Proof. Consider first mapping patterns having exactly n_k cycles of length k. Harary [5] observed that the coefficient of x^n in $Z(S_{n_k}, Z(C_k, T(x)))$, or, which is the same, the coefficient of $x^n y_k^{n_k}$ in $Z(S_{n_k}, Z(C_k, T(x)))y_k^{n_k}$ is equal to the number of mapping patterns with n_k cycles of length k having n points. Denote by M_n $(n_1, n_2, \ldots)$ the number of all mapping patterns on n points having cyclic structure $(n_1, n_2, \ldots)$ (n_1 cycles of length 1, n_2 cycles of length 2 and so on). Obviously,

$$(2.3) \qquad E^{(p_n)}\Big(\prod_{k=1}^{n} y_k^{\alpha_k}\Big) = \frac{1}{p_n} \sum_{\substack{(n_1, n_2, \ldots) \\ \sum n_k \leqq n}} M(n_1, n_2, \ldots) y_1^{n_1} y_2^{n_2} \ldots .$$

Let $x^n[F(x)]$ denote the coeficient of x^n in any power series $F(x)$. Applying the same reasonings as in Step 3 of Harary's paper [5] and relations (2.2) and (2.3), we obtain that

$$p_n E^{(p_n)}\Big(\prod_{k=1}^{n} y_k^{\alpha_k}\Big)$$

$$= x^n\Big[\sum_{\text{all } n_k=0}^{\infty} \prod_{k=1}^{\infty} Z(S_{n_k}, Z(C_k, T(x))y_k^{n_k}\Big]$$

$$= x^n\Big[\prod_{k=1}^{\infty} \Big(1 + \sum_{k=1}^{\infty} Z(S_r, Z(C_k, T(x)))y_k^r\Big]$$

$$= x^n\Big[\prod_{k=1}^{\infty} \exp \sum_{m=1}^{\infty} \frac{y_k^m}{m} Z(C_k, T(x^m))\Big],$$

which imply the required result.

To prove (ii) we shall note that from the definition of the numbers c_k (the number of connected mapping patterns on k points) and from Pólya's enumeration theorem it follows that $x^n y_k^{n_k}[Z(S_{n_k}, c_k x^k) y_k^{n_k}]$ is the number of mapping patterns on n points with n_k components of size k. Then, applying the same reasonings as in the proof of (i), we obtain (ii). ∎

We need also some properties of the generating function $T(x) = \sum_{n=1}^{\infty} t_n x^n$ for the rooted unlabelled trees. Otter [13] (see also [6; §9.5]) showed that it has radius of convergence

$$(2.4) \qquad \varrho = .3383$$

and that $T(x)$ has an expansion about $x = \varrho$ of the form

$$(2.5) \qquad T(x) = 1 - b(\varrho - x)^{1/2} + b_2(\varrho - x) + b_3(\varrho - x)^{3/2} + \ldots, \quad b = 2.6811 \ldots .$$

Since $t_n>0$, $x=\varrho$ is the only singularity on the circle of convergence of $T(x)$. Meir and Moon [11; p. 63] summarized four important properties of $T(x)$, which are straightforward consequences of Otter's results, in a separate lemma. We shall formulate it below adding a fifth property about the possibility of extending the region in which $T(x)$ is analytic (see [13; p. 593]).

Lemma 2. (i) *$T(x)$ is analytic for $|x|\leqq\varrho$, $x\neq\varrho$; furthermore, $|T(x)|<1$ if $|x|\leqq\varrho$, $x\neq\varrho$.*

(ii) *$|T(x^j)|\leqq T(\varrho^j)$ for $|x|\leqq\varrho$ and $j=2,3,\ldots$.*

(iii) *$T(x^j)$ is analytic for $|x|<\varrho^{1/2}$ and $j=2,3,\ldots$.*

(iv) *If $|x|\leqq\varrho$, then $|T(x)/x|\leqq T(\varrho)/\varrho=1/\varrho$; in particular, $T(\varrho^j)\leqq\varrho^{j-1}$ for $j=1,2,\ldots$.*

(v) *The region in which $T(x)$ is analytic may be extended to a circular region of radius larger then ϱ, provided we make a cut in this region extending along the positive reals from $x=\varrho$.* ▮

Let $C(x)=\sum_{n=1}^{\infty}c_nx^n$ denote the generating function for the numbers c_n of connected mapping patterns. It follows immediately from the Harary's result [5] for the generating function of connected mapping patterns in which the cycle has length k that

$$C(x)=\sum_{k=1}^{\infty}Z(C_k,T(x)).$$

The following two expressions for $P(x)$ and $C(x)$, due to Meir and Moon [11; p. 64 and 65], are appropriate for asymptotic analysis.

Lemma 3. (i) $C(x)=-\sum_{d=1}^{\infty}\frac{\varphi(d)}{d}\log(1-T(x))$.

(ii)
$$P(x)=\prod_{j=1}^{\infty}(1-T(x^j))^{-1}.\quad ▮$$

Using Lemmas 2 and 3, Meir and Moon [11; p. 64 and 66] determined the asymptotics of the coefficients c_n and p_n.

Lemma 4. (i) $c_n=\varrho^{-n}/2n+O(\varrho^{-n}n^{3/2})$.

(ii)
$$p_n=\frac{P^*(\varrho)}{b}\varrho^{-n}(\varrho\pi n)^{-1/2}+O(\varrho^{-n}n^{-3/2})$$

where $P^*(x)=\prod_{j=2}^{\infty}(1-T(x^j))^{-1}$. ▮

Seeking asymptotics of coefficients of generating functions we shall use a simple lemma whose proof is trivial and may be found in [1; p. 496].

Lemma 5. *Suppose that $A(z)=\sum_{n=0}^{\infty}a_nz^n$ and $B(z)=\sum_{n=0}^{\infty}b_nz^n$ are power series with radii of convergence $\alpha>\beta\geqq 0$, respectively, and $b_{n-1}/b_n\to q=\text{const}$, as $n\to\infty$. If $A(q)\neq 0$, then $c_n/b_n\to A(q)$ as $n\to\infty$, where $\sum_{n=0}^{\infty}c_nz^n=A(z)B(z)$.* ▮

In Section 5 we shall need the following lemma, due to Darboux [3; p. 20] (see also [1; pp. 499—502]):

Lemma 6. *Suppose the function* $g(x)=\sum_{n=0}^{\infty} g_n x^n$ *has a non-zero finite radius of convergence R and that* $x=R$ *is the only singularity on the circle of convergence. If* $g(x)$ *has an expansion about* $x=R$ *of the form*

$$g(x) = (R-x)^{-s}G(x)+H(x),$$

where $G(x)$ *and* $H(x)$ *are analytic for* $|x|\leqq R$, $G(R)\neq 0$ *and* $s\neq 0, -1, -2, \ldots,$ *then*

$$g_n = \frac{G(R)}{\Gamma(s)} R^{-n-s}n^{s-1}+O(R^{-n}n^{s-2}),$$

as $n\to\infty$, *where* $\Gamma(s)$ *is the Euler gamma function.* ∎

3. The number of cyclic points

Let λ be the number of the cyclic points in a random mapping pattern. The following statement describes the asymptotic distribution of λ with respect to the measure $P^{(p_n)}$ as $n\to\infty$.

Theorem 1. *If* $n\to\infty$ *and* $u=ln^{-1/2}$, *where l is a positive integer, then*

$$P^{(p_n)}\{\lambda n^{-1/2} = u\} = \left(\frac{b^2\varrho}{2} n^{-1/2}u \exp(-b^2\varrho^{u^2/4})\right)(1+o(1))$$

for all $u=o(n^{1/2}/\log n)$.

Proof. Since $\lambda=\sum k\alpha_k$, we subsstitute $y_k=y^k$ in the relation of Lemma 1(i). Then, its left-hand side changes into

$$\prod_{k=1}^{\infty}\exp\sum_{m=1}^{\infty}\frac{y^{km}}{m}Z(C_k, T(x^m)) = \exp\sum_{m=1}^{\infty}\frac{1}{m}\sum_{k=1}^{\infty}(y^m)^k Z(C_k, T(x^m)).$$

Meir and Moon [11; pp. 66—67] showed that the last expression is equal to $\prod_{j=1}^{\infty}(1-y^jT(x^j))^{-1}$. Thus, Lemma 1(i) becomes

$$\prod_{j=1}^{\infty}(1-y^jT(x^j))^{-1} = 1+\sum_{n=1}^{\infty}p_n x^n E^{(p_n)}(y^\lambda). \tag{3.1}$$

Using that $E^{(p_n)}(y^\lambda)=\sum_{k=1}^{n}P^{(p_n)}\{\lambda=k\}y^k$ and interchanging the order of summation, we can rewrite (3.1) in the following way:

$$P^*(x,y)(1-yT(x))^{-1} = 1+\sum_{k=1}^{\infty}y^k\sum_{n=k}^{\infty}p_nP^{(p_n)}\{\lambda = k\}x^n, \tag{3.2}$$

where $P^*(x,y)=\prod_{j=2}^{\infty}(1-y^jT(x^j))^{-1}$. Let $A_l(x)=y^l[P^*(x,y)(1-yT(x))^{-1}]$. From (3.2) it follows that

$$A_l(x) = \sum_{n=l}^{\infty} p_n P^{(p_n)}\{\lambda = l\} x^n. \tag{3.3}$$

It is easy to show, using Lemma 2(i)—(iv), that the radius of convergence of $P^*(x,y)$ is greater than that of $(1-yT(x))^{-1}$ (both expansions are taken with respect to y). Thus, we can apply the result of Lemma 5 to the power series $P^*(x,y)$ and $(1-yT(x))^{-1}$ to obtain

$$A_l(x) \sim P^*(x, 1/T(x))T^l(x), \quad l \to \infty. \tag{3.4}$$

Now we wish to estimate $x^n[T^l(x)]$ as $n\to\infty$, setting $l=un^{1/2}$, $0<u=o(n^{1/2}/\log n)$. By Cauchy's integral theorem

$$x^n[T^l(x)] = \frac{1}{2\pi i}\int_\gamma T^l(x)\frac{dx}{x^{n+1}}. \tag{3.5}$$

According to Lemma 2(v) we can choose for the path of integration γ the boundary of the circle $|x|\leqq \varrho+(\log n)/n$ with the radial cut $(\varrho, \varrho+(\log n)/n)$ assuming that $(\varrho-x)^{1/2}>0$ for $0<x<\varrho$ and that n is suffisiently large. We have

$$T(x) = 1-(\varrho-x)^{1/2}(b-L(x)),\ L(x) = b_2(\varrho-x)^{1/2}+b_3(\varrho-x)+\dots \tag{3.6}$$

(sse (2.5)) and

(3.7)

$$\begin{cases}(\varrho-x)^{1/2} = (x-\varrho)^{1/2}e^{-i\pi/2} = -i(x-\varrho)^{1/2} & \text{along the upper shore of the cut;}\\ (\varrho-x)^{1/2} = (x-\varrho)^{1/2}e^{i\pi/2} = i(x-\varrho)^{1/2} & \text{along the lower shore of the cut.}\end{cases}$$

Since the integral along the boundary of the circle is $O((\varrho+(\log n)/n)^{-n})$, using (3.5)—(3.7), we find that

$$x^n[T^l(x)] = \frac{1}{2\pi i}\int_{\varrho}^{\varrho+(\log n)/n}((1+i(x-\varrho)^{1/2}(b-L(x)))^l - \tag{3.8}$$

$$-(1-i(x-\varrho)^{1/2}(b-L(x)))^l)\frac{dx}{x^{n+1}}+O((\varrho+(\log n)/n)^{-n}).$$

Now we substitute $x-\varrho=t/n$ in the integral of the right-hand side of (3.8). It is easy to check that the following asymptotic relations hold:

$$(1\pm i(x-\varrho)^{1/2}(b-L(x)))^{un^{1/2}} = \exp(\pm iubt^{1/2}+O(un^{-1/2}\log n)) =$$

$$= \exp(\pm iubt^{1/2}+o(1)),$$

$$(1+t/n\varrho)^{-n} = e^{-t/\varrho}(1+O(t^2/n)) = e^{-t/\varrho}(1+O((\log^2 n)/n)),$$

$$O((\varrho+(\log n)/n)^{-n}) = o(\varrho^{-n}/n)$$

for $0<t<\log n$. Thus, after simple manipulations we obtain that

$$x^n[T^l(x)] \sim \frac{\varrho^{-n-1}}{2\pi i n} \int_0^\infty (e^{iubt^{1/2}} - e^{-iubt^{1/2}}) e^{-t/\varrho}\, dt$$

$$(3.9) \qquad = \frac{\varrho^{-n-1}}{\pi n} \int_0^\infty e^{-t/\varrho} \sin(but^{1/2})\, dt =$$

$$= \frac{bu}{2n} \varrho^{1/2} \pi^{-1/2} \varrho^{-n} \exp(-b^2 \varrho u^2/4).$$

Furthermore (3.9) yields the convergence $x^{n-1}[T^l(x)]/x^n[T^l(x)] \to \varrho$, $n \to \infty$ for $l = un^{1/2}$ and $u = o(n^{1/2}/\log n)$. Hence, applying again the result of Lemma 5 under the same assumptions on l, u and n, we get

$$(3.10) \qquad x^n\big[P^*(x, 1/T(x))T^l(x)\big] \sim P^*(\varrho) x^n[T^l(x)]$$

where $P^*(\varrho) = P^*(\varrho, 1/T(\varrho)) = P^*(\varrho, 1)$ (for the definition of $P^*(x)$ see also Lemma 4(ii)). It is easy to show now that from (3.3), (3.4), (3.9) and (3.10) it follows that

$$p_n P^{(\mathscr{P}_n)}\{\lambda n^{-1/2} = u\} \sim \frac{bu}{2n} \varrho^{-n} \pi^{-1/2} \varrho^{1/2} P^*(\varrho) \exp(-b^2 \varrho u^2/4).$$

This and Lemma 4(ii) imply the required result. ∎

Theorem 2. *If $n \to \infty$ and $u = ln^{-1/2}$, where l is a positive integer, then*

$$P^{(c_n)}\{\lambda n^{-1/2} = u\} = \big(b(\varrho/\pi n)^{1/2} \exp(-b^2 \varrho u^2/4)\big)\big(1 + o(1)\big)$$

for all $u = o(n^{1/2}/\log n)$.

Proof. Harary [5] established that the coefficient of x^n in $Z(C_l, T(x))$ (see (2.1)) is the number of connected mapping patterns on n points whose cycle has length l. In other words,

$$(3.11) \qquad x^n\big[Z(C_l, T(x))\big] = c_n P^{(c_n)}\{\lambda = l\}.$$

From (2.1) it follows that $Z(C_l, T(x))$ can be represented in the form

$$(3.12) \qquad Z(C_l, T(x)) = (T^l(x)/l) + U(x),$$

where

$$(3.13) \qquad U(x) = \frac{1}{l} \sum_{\substack{d|l \\ d \neq 1}} \varphi(d) (T(x^d))^{l/d}.$$

Lemma 2(iii) yields that $U(x)$ is analytic for $|x| < \varrho + \varepsilon$ for a suitable $\varepsilon > 0$. Applying again Cauchy's integral theorem with the same path of integration γ and taking into account (3.11) and (3.12), we obtain that

$$(3.14) \qquad c_n P^{(c_n)}\{\lambda = l\} = \frac{1}{2\pi i l} \int_\gamma T^l(x) \frac{dx}{x^{n+1}} + \frac{1}{2\pi i} \int_\gamma U(x) \frac{dx}{x^{n+1}}.$$

The arguments, used in the proof of Theorem 1, yield that

$$(3.15)\qquad \frac{1}{2\pi i l}\int_{\gamma} T^{l}(x)\frac{dx}{x^{n+1}} = \frac{b}{2}\pi^{1/2}\varrho^{1/2}\varrho^{-n}\exp(-b^2\varrho u^2/4)+o(\varrho^{-n}n^{-3/2}).$$

Furthermore, since $U(x)$ is analytic in a region larger than $|x|\leqq\varrho$, it is easy to check that

$$(3.16)\qquad \int_{\gamma} U(x)\frac{dx}{x^{n+1}} = o(\varrho^{-n}n^{-3/2}).$$

Now a combination of (3.14)—(3.16) and the result of Lemma 4(i) prove our assertion. ∎

4. Cycles and components with fixed number of points

Since the methods of this section will be similar to those used in the previous one, we shall omit some of the details. First, we consider the asymptotic behaviour of α_j (the number of cycles of length j in a random mapping pattern) for fixed j.

Theorem 3. $\lim_{n\to\infty} E^{(p_n)}(y^{\alpha_j})=f_j(y)/f_j(1)$, $j=1,2,\dots$, *where* $f_j(y)=\exp\sum_{m=1}^{\infty} d_m(j)y^m/m$ *and* $d_m(j)=Z(C_j, T(\varrho^m))$, $j=1,2,\dots$.

Proof. First note that $Z(S_m, C(x))$ is the generating function for mapping patterns whose graph has m components (cycles) (see [5]). Combinating this fact with (2.1) it can be easily verified (see [11; p. 66) that

$$(4.1)\qquad P(x) = \exp\sum_{m=1}^{\infty} C(x^m)/m.$$

Further, setting $y_j=y$ and $y_k=1$ for $k\neq j$ in Lemma 1(i) and applying (4.1), we obtain

$$1+\sum_{n=1}^{\infty} p_n x^n E^{(p_n)}(y^{\alpha_j}) = P(x)\exp\sum_{m=1}^{\infty}\frac{y^m-1}{m}Z(C_j, T(x^m))$$

$$= (1-T(x))^{-1}V(x)\exp\left(\frac{y-1}{j}T^j(x)\right),$$

where

$$V(x) = P^*(x)\exp\left(\frac{y-1}{j}\sum_{\substack{d|j\\ d\neq 1}}\varphi(d)(T(x^d))^{j/d}\right)+$$

$$+\sum_{m=2}^{\infty}\frac{y^m-1}{m}Z(C_j, T(x^m)).$$

The quantity $x^n\left[(1-T(x))^{-1}\exp\left(\frac{y-1}{j}T^j(x)\right)\right]$ can be estimated in the same way as in the proof of Theorem 1, whence we obtain

$$x^n\left[(1-T(x))^{-1}\exp\left(\frac{y-1}{j}T^j(x)\right)\sim\frac{e^{(y-1)/j}}{b}\varrho^{-n}(\varrho\pi n)^{-1/2},\right.$$

so that

$$x^{n-1}\left[(1-T(x))^{-1}\exp\left(\frac{y-1}{j}T^j(x)\right)\right]\Big/x^n\left[(1-T(x))^{-1}\exp\left(\frac{y-1}{j}T^j(x)\right)\right]\to\varrho$$

as $n\to\infty$. Applying Lemmas 4(ii) and 5, we get the required asymptotic result. ∎

We continue with the characteristic β_j, the number of components on j points in a random mapping pattern.

Theorem 4. $\lim_{n\to\infty} E^{(p_n)}(y^{(\beta_j)})=\left(\frac{1-\varrho}{1-\varrho^j y}\right)^{e_j}$, $j=1, 2, \ldots.$

Proof. After the substitution $y_j=y$ and $y_k=1$ for $k\neq j$ in Lemma 1(ii), using identity (4.1), we immediately obtain that

$$1+\sum_{n=1}^{\infty}p_n x^n E^{(p_n)}(y^{\beta_j}) = P(x)\left(\frac{1-x^j}{1-x^j y}\right)^{e_j}.$$

Since $\lim_{n\to\infty} p_{n-1}/p_4=\varrho$ (see Lemma 4(iii)), applying directly Lemma 5, we find that the required relation holds. ∎

5. The number of components

In order to determine the limit distribution of the number $\varkappa=\Sigma\beta_j=\Sigma\alpha_j$ of components of a random mapping pattern on n points, as $n\to\infty$, we shall use characteristic functions.

Theorem 5. *If* $\varkappa'=(\varkappa-(1/2)\log n)/((1/2)\log n)^{1/2}$, *then*

$$\lim_{n\to\infty} P^{(p_n)}\{\varkappa'<u\} = (2\pi)^{-1/2}\int_{-\infty}^{u} e^{-v^2/2}\,dv.$$

Proof. In a familiar way (see also Theorem 5 of [11]) we obtain

$$1+\sum_{n=1}^{\infty}p_n x^n E^{(p_n)}(y^{\varkappa}) = \exp\left(yC(x)+W(y,x)\right), \tag{5.1}$$

where $W(y,x)=\sum_{m=2}^{\infty}y^m C(x^m)/m$ is analytic for $|x|<\varrho+\varepsilon$ for some $\varepsilon>0$. By Lemma 3(i) and (3.6), (5.1) becomes

$$1+\sum_{n=1}^{\infty}p_n x^n E^{(p_n)}(y^{\varkappa}) = (1-T(x))^{-y}\exp\left(yD(x)+W(y,x)\right) =$$
$$= (\varrho-x)^{-y/2}(b-L(x))^{-y}\exp\left(yD(x)+W(y,x)\right),$$

where $D(x)=-\sum_{d=2}^{\infty}\varphi(d)\log(1-T(x^d))/d$ is also analytic for $|x|<\varrho+\varepsilon$. Hence

$$p_n E^{(p_n)}(y^{\varkappa}) = \frac{\exp(yD(\varrho)+W(y,\varrho))}{b^y\Gamma(y/2)}\varrho^{-n-y/2}n^{(y/2)-1}+O(\varrho^{-n}n^{(y/2)-2})$$

by Lemma 6. Setting $y=\exp(it/((1/2)\log n)^{1/2})$ and applying Lemma 4(ii), after some cancellations, we obtain

$$E^{(p_n)}(e^{it\varkappa'})\sim\exp\left[-it\left(\frac{1}{2}\log n\right)^{1/2}+\left(\frac{1}{2}\log n\right)(e^{it/\left(\frac{1}{2}\log n\right)^{1/2}}-1)\right]=$$

$$=\exp(-(t^2/2)+O((\log n)^{-1/2}))\to e^{-t^2/2},$$

as $n\to\infty$, which completes the proof. ∎

6. Concluding remarks

From Theorem 1 it follows that the asymptotic density function of the number λ of cyclic points in a random mapping pattern, normalized by $n^{-1/2}$ is $(b^2\varrho/2)u\exp(-b^2\varrho u^2/4)$ with respect to the measure $P^{(p_n)}$. Meir and Moon [11] have have shown that $E^{(p_n)}(\lambda)=(1/b)(\pi n/\varrho)^{1/2}+O(1)\sim(1.136\ldots)n^{1/2}$. Harris [7] (see also [9; Chap. III, §1]) has proved that the asymptotic density of the same characteristic for the graph of a random mapping on n labelled points is $ue^{-u^2/2}$. Theorem 2 implies that the asymptotic density of $\lambda n^{-1/2}$ with respect to the measure $P^{(c_n)}$ coincides with the density of the random variable $|\xi|$, where ξ has normal distribution with mean 0 and variance $2/b^2\varrho$. Rényi [15] proved a corresponding result for random mappings on labelled point-sets when the parameters of ξ are 0 and 1. In addition, Meir and Moon [11] have derived that $E^{(c_n)}(\lambda)=(2/b)(n/\pi\varrho)^{1/2}+O(1)\sim(.723\ldots)n^{1/2}$.

For the characteristics α_j and β_j (see Section 4, Theorems 3 and 4) we derived the asymptotics of their generating functions in the form $f_j(y)/f_j(1)$, where the $f_j(y)$ are explicitly given. The corresponding limit distributions of α_j and β_j for the graph of a random mapping on n labelled points were found by Stepanov [16] and Kolchin [8] (see also [9; Chap. I, § 12]), respectively.

Meir and Moon [11] have shown that the expected number of components $E^{(p_n)}(\varkappa)$ of random mapping patterns on n points is $\sim(1/2)\log n$ which coincides with the corresponding result for random mappings on n labelled points (see [10]). Theorem 5 asserts that $(\varkappa-(1/2)\log n)((1/2)\log n)^{-1/2}$ is assymptotically standard normal as $n\to\infty$. Stepanov [16] (see also [9; Chap. I, § 12]) has proved the same result for random mappings with labelled point-sets.

The comparison of the results for labelled and unlabelled graphs of finite mappings was given in a tabular form in Table 1.

References

[1] E. A. Bender, Asymptotic methods in enumeration, *SIAM Review* **16** (1974), 485—515.
[2] N. G. de Bruijn and D. A. Klarner, Multisets of aperiodic cycles, *SIAM J. Alg. Disc. Meth.* 3 (1982), 359—368.
[3] G. Darboux, Mémoire sur l'approximation des fonctions de trés grands nombres, et sur une classe étendu développments en série, *J. Math. Pures et Appliquées* 4 (1978), 5—56.
[4] M. A. Evgraphov, *Asymptotic Estimates and Entire Functions,* Nauka, Moscow, 1979 (in Russian).
[5] F. Harary, The number of functional digraphs, *Math. Ann.* **138** (1959), 203—210.
[6] F. Harary and E. Palmer, *Graphical Enumeration,* Academic Press, New York, 1973.
[7] B. Harris, Probability distributions related to random mappings, *Ann. Math. Statist.* **31** (1960), 1045—1062.
[8] V. F. Kolchin, A problem of the allocation of particles in cells and random mappings, *Theory Probability Appl.* **21** (1976), 48—63.
[9] V. F. Kolchin, *Random Mappings, Optimization Software,* Inc., New York, 1986.
[10] M. D. Kruskal, The expected number of components under a random mapping function, *Amer. Math. Monthly* **61** (1954), 392—397.
[11] A. Meir and J. W. Moon, On random mapping patterns, *Combinatorica* **4** (1984), 61—70.
[12] L. Mutafchiev, *On some stochastic problems of discrete mathematics, Mathematics and Mathematical Education* (Sunny Beach, 1984), 57—80, B"lgar. Akad. Nauk, Sofia, 1984.
[13] R. Otter, The number of trees, *Ann. Math.* **49** (1948), 583—599.
[14] R. C. Read, A note on the number of functional digraphs, *Math. Ann.* **143** (1961), 109—110.
[15] A. Rényi, O connected graphs, I, *Publ. Math. Inst. Hung. Acad. Sci.* **4** (1959), 385—388.
[16] V. E. Stepanov, Limit distributions of certain characteristics of random mappings, *Theory Probability Appl.* **14** (1969), 612—626.

Ljuben R. Mutafchiev

Institute of Mathematics with Computer Center
Bulgarian Academy of Sciences